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Anti-*BZ*-Structure in Effect Algebras

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The definitions of sharply approximating effect algebras, anti-*BZ*-effect algebras, central approximating effect algebras, and S-anti-*BZ*-effect algebras are given, the relationships between sharply approximating effect algebras and anti-*BZ*-effect algebras, between central approximating effect algebras and anti-*BZ*-effect algebras are established, and the set of anti-*BZ*-sharp elements in S-anti-*BZ*-effect algebras is proved to be an orthomodular lattice.

KEY WORDS: sharply approximating effect algebras; anti-*BZ*-effect algebras; central approximating effect algebras; S-anti-*BZ*-effect algebras.

1. INTRODUCTION AND BASIC DEFINITIONS

Since in 1936 Birkhoff and von Neumann regarded the lattice of all closed subspaces of a separable infinite dimensional Hilbert space which is an orthomodular lattice as a proposition system for a quantum mechanical entity (R. Miklós, 1998), orthomodular lattices have been considered as a mathematical model for a calculus of quantum logic. With the development of the theory of quantum logics, effect algebras as a quantum structure which generalize orthomodular lattices, orthomodular posets, and orthoalgebras, are also regarded as a mathematical model of quantum logics (Foulis *et al.*, 1992). The main advantage of an effect algebra is that it can embody sharp or unsharp properties (Lahti and Maczynski, 1995). However, the shortcoming of it is that the set of sharp elements in a general effect algebra is not an orthomodular lattice, not even an orthoalgebra, which cannot meet the need of physical relevant systems. To avoid the shortcoming, a BZ-structure was introduced and some properties were obtained (Cattaneo, 1997; Cattaneo and Nistico, 1989; Gudder, 1998a). For the same reason, in this note, we introduce an anti-*BZ*-structure in effect algebras and get some good

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properties. We give the definitions of sharply approximating effect algebras, anti-*BZ*-effect algebras, central approximating effect algebras, and S-anti-*BZ*-effect algebras establish the relationship between sharply approximating effect algebras; and anti-*BZ*-effect algebras, the relationship between central approximating effect algebras and anti-*BZ*-effect algebras; and prove the set of anti-*BZ*-sharp elements in S-anti-*BZ*-effect algebras is an orthomodular lattice. Since an S-dominating effect algebra (Gudder, 1998b) is an S-anti-*BZ*-effect algebra, we conclude that S-anti-*BZ*-effect algebras may be an abstract model for quantum logics in some sense.

Definition 1.1. (Cattaneo and Nistico, 1989). Let $(P, \leq, 0, 1,')$ be a De Morgan Poset. A *B*-complementation on P is a unary operation $\sim: P \to P$ that satisfies: $a \leq a^{\sim\sim}$, $a \leq b \Rightarrow b^{\sim} \leq a^{\sim}$, $a \wedge a^{\sim} = 0$, and $a^{\sim'} = a^{\sim\sim}$. If $\sim$ is a *B*-complementation on P, we call $(P, \leq, 0, 1, ', \sim)$ a *BZ*-Poset.

Definition 1.2. (Foulis *et al.*, 1992). A structure $(P, \oplus, 0, 1)$ is called an effect algebra if 0,1 are two distinguished elements and $\oplus$ is a partially defined binary operation on P that satisfies the following conditions for any $a, b, c \in P$:

(E1) $b \oplus a = a \oplus b$.
(E2) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.
(E3) For every $a \in P$, there exists a unique $b \in P$ such that $a \oplus b = 1$ (we put $a' = b$).
(E4) If $1 \oplus a$ is defined, then $a = 0$.
An orthoalgebra is an algebraic system $(P, 0, 1, \oplus)$ that satisfies (E1), (E2), (E3), and
(E5): If a $\oplus$ a exists, then $a = 0$.

Remark 1.3. Let a and b be two elements of an effect algebra P.

(i) $a \perp b$ iff $a \leq b'$ iff $a \oplus b$ is defined in P.
(ii) $a \leq b$ iff there exists an element $c \in P$ such that $a \oplus c = b$.
(iii) b is the orthocomplement of a iff b is a unique element of P such that $a \oplus b = 1$ and it is written a'.

Obviously, $(P, \leq, 0, 1, ')$ is a De Morgan poset.

Definition 1.4. (Gudder, 1998a). Let $(P, \leq, 0, 1,')$ be a De Morgan Poset. An element $a \in P$ is sharp if $a \wedge a' = 0$. Put $P_s = \{a \in P | a \wedge a' = 0\}$. Obviously, 0, 1, $\in P_s$.

Example 1.5. (Lahti and Maczynski, 1995). Let H be a complex Hilbert space and let $\varepsilon(H)$ be the set of self-adjoint linear operators on H, whose inner product is $\langle | \rangle$ and satisfies $\forall \phi \in H$, $0 \leq \langle A\phi|\phi\rangle \leq \|\phi\|^2 \cdot \varepsilon(H)$ is a poset with respect to the partial ordering $A_1 \leq A_2$ iff $\forall \phi \in H$, $\langle A_1\phi|\phi\rangle \leq \langle A_2\phi|\phi\rangle$. Obviously, 0 is the smallest element, 1 is the largest element on $\varepsilon(H)$. For $A, B \in \varepsilon(H)$, we write $A \perp B$ if $A + B \in \varepsilon(H)$ and we define $A \oplus B = A + B$. If we define $A' = 1 - A$ for $A \in \varepsilon(H)$, it is clear that $(\varepsilon(H), \oplus, 0, 1)$ is an effect algebra which we call a Hilbert space effect algebra. The family $\varepsilon(H)_s$ of all sharp elements is the set $\mathcal{P}(H)$ of all orthogonal projections on H.

Definition 1.6. (Gudder, 1998a). Let $(P, \leq 0, 1, ')$ be a De Morgan Poset, P is sharply dominating if every $a \in P$ is dominated by a smallest sharp element $\mu(a) \in P_s$ (i.e., (i) $a \leq \mu(a)$, (ii) if $a \leq b \in P_s$, then $\mu(a) \leq b$).

Definition 1.7. (Gudder, 1998b). A sharply dominating De Morgan poset P is called S-dominating DM-Poset if $a \wedge p$ exists for every $a \in P$, $p \in P_s$.

Lemma. 1.8. (Gudder, 1998a). *Let P be an effect algebra and $a \in P_s$. If $b \in P$ with $a \perp b$ then $a \oplus b$ is a minimal upper bound for a and b.*

Lemma. 1.9. (Foulis *et al.*, 1992). *An orthomodular poset is an orthalgebra P that satisfies the following conditions:*

For $p, q \in P$, if $p \perp q$, then $p \vee q$ exists and $p \vee q = p \oplus q$.

Definition 1.10. (Greechie *et al.*, 1995). For an effect algebra $(P, \oplus, 0, 1)$, an element $z \in P$ is called central iff for every $x \in P$ there exist $x \wedge z$ and $x \wedge z'$ and $x = (x \wedge z) \vee (x \wedge z')$. The set $C(P)$ of all central elements of P is called the center of P.

2. ANTI-*BZ*-EFFECT ALGEBRAS

Definition 2.1. Let P be a De Morgan Poset. A unary operation $* : P \longrightarrow P$ is called an anti-*BZ*-complementation if it satisfies

(i) $a^{**} \leq a$.
(ii) $a \leq b \Rightarrow b^* \leq a^*$.
(iii) $a \vee a^* = 1$.
(iv) $a^{*'} = a^{**}$.

If * is an anti-B-complementation on P, we call $(P, \leq, 0, 1, ^*, ')$ an anti-*BZ*-Poset.

Example 2.2. The poset $(\varepsilon(H), 0, 1, *, ')$ is an anti-*BZ*-Poset with respect to

(i) $F' = 1 - F$, for all $F \in \varepsilon(H)$; and

(ii) $F^* = E_{\text{Ker}(1-F)^\perp} = E_{\overline{\text{Ran}(1-F)}}$. Where Ker($F$) denotes the kernel of the operator F and $\overline{\text{Ran}(F)}$ denotes the closure of the Ran(F) of F, E_M denotes the projection onto the subspace M of H.

An element a in an anti-BZ-Poset P is anti-BZ-sharp relative to $*$ if $a = a^{**}$. Put $P_s^* = \{a \in P | a = a^{**}\}$. Obviously, $\varepsilon(H)_s^* = P(H)$.

It is easy to prove the following proposition.

Proposition 2.3. *Let P be an anti-BZ-Poset. Then unary operation $**$: $P \to P$ satisfies the following properties (where we define $0^* = 1$).*

(i) $1^{**} = 1$.
(ii) $a^{**} \leq a$, *for all* $a \in P$.
(iii) $a^{****} = a^{**}$, *for all* $a \in P$.
(iv) $a \leq b \Rightarrow a^{**} \leq b^{**}$, *for* $a, b \in P$.
(v) $((a^{**})')^{**} = (a^{**})'$, *for all* $a \in P$.
(vi) $a^{**} \wedge (a^{**})' = 0$, *for all* $a \in P$.
(vii) $a \vee (a^{**})' = 1$, *for all* $a \in P$.

Obviously, the operator $**$ is an interior operator by (i), (ii), (iii), and (iv). Together with (v), we conclude that it is a universal quantifer (Halmos, 1962).

Conversely, we can construct an anti-BZ-poset by (i), (ii), (iv), (v), and (vii).

Proposition 2.4. *Let $(P, \leq, 0, 1, ')$ be a De Morgan Poset with the mapping η: $P \to P$ satisfying the following conditions:*

(i) $\eta(1) = 1$.
(ii) $\eta(a) \leq a$, *for all* $a \in P$.
(iii) $a \leq b \Rightarrow \eta(a) \leq \eta(b)$, *for* $a, b \in P$.
(iv) $\eta(\eta(a)') = \eta(a)'$, *for all* $a \in P$.
(v) $a \vee \eta(a)' = 1$, *for all* $a \in P$.

Then $(P, \leq, 0, 1, ', \eta)$ is an anti-BZ-Poset, with respect to $a^* = \eta(a)'$.

Proof: Let $a, b \in P$, $a \leq b$, then $\eta(a) \leq \eta(\text{b})$ by (iii). So $\eta(b)' \leq \eta(a)'$, i.e., $b^* \leq a^*$. Since $a^{**} = \eta(\eta(a)')' = \eta(a)$, then $a^{**} \leq a$ by (ii). Obviously, $a \vee a^* = a \vee \eta(a)' = 1$. $a^{**} = \eta(a) = a^{*\prime}$. Thus, $(P, \leq, 0, 1, ', \eta)$ is an anti-BZ-Poset. □

Proposition 2.5. *Let P be an anti-BZ-Poset. Then $a \in P_s^*$ if and only if $a' = a^*$.*

Remark 2.6. If P is an anti-BZ-Poset, then $P_s^* \subseteq P_s$.

However, in general, $P_s^* \neq P_s$. For instance, let P be an orthoalgebra, define $1^* = 0$, and for all $p \neq 0$, let $p^* = 1$. Evidently, $(P, \leq, 0, 1, *, ')$ is an anti-BZ Poset. But $P_s^* = \{0, 1\}$, $P_s = P$.

Lemma. 2.7. *Let P be an anti-BZ-Poset.*

(i) If $a \wedge b$ exists in P, then $(a \wedge b)^* = a^* \vee b^*$.
(ii) If $P_s^* = P_s$, then the following statements are equivalent:
(1) $a \in P_s$, (2) $a = a^{**}$, (3) $a' = a^*$.

Corollary 2.8. *Let P be an anti-BZ-Poset with $P_s^* = P_s$ and let $a, b \in P_s$.*

(i) If $a \wedge b$ exists, then $a \wedge b \in P_s$.
(ii) If $a \vee b$ exists, then $a \vee b \in P_s$.

Proof: (i) Suppose that $a \wedge b$ exists. Applying Lemma 2.7(i), we have $(a \wedge b)^* = a^* \vee b^* = a' \vee b' = (a \wedge b)'$. By Lemma 2.7(ii), we have that $a \wedge b \in P_s^* = P_s$.
(ii) By $(a \vee b)' = a' \wedge b'$. □

Definition 2.9. Let $(P, \oplus, 0, 1)$ be an effect algebra. P is sharply approximating if for every $a \in P$ is approximated by a largest sharp element $\nu(a) \in P_s$ (i.e., (i) $\nu(a) \leq a$, (ii) if $b \leq a$ and $b \in P_s$, then $b \leq \nu(a)$).

Remark 2.10. Let $(P, \oplus, 0, 1)$ be an effect algebra. If $a \in P_s$, then $a = \nu(a) = \mu(a)$, $a' = \nu(a') = \mu(a')$.

Example 2.11. $\varepsilon(H)$ is a sharply approximating effect algebra. Indeed, for all $F \in \varepsilon(H)$, $\nu(F) = E_{\mathbf{Ker}(1-F)}$. Obviously, $\nu(F) \in \mathcal{P}(H)$, and since for all $x \in H$, $\langle \nu(F)x|x\rangle = \|x_1\|^2 \leq \langle Fx|x\rangle = \|x_1\|^2 + \langle Fx_2|_2\rangle$, where $x_1 \in \mathrm{Ker}(1-F)$, $x_2 \in \mathrm{Ker}(1-F)^\perp$, which implies $\nu(F) \leq F$. If $G \in \mathcal{P}(H)$, and $G \leq F$, then $1 - F \leq 1 - G$, so $\mathrm{Ker}(1-G) \leq \mathrm{Ker}(1-F)$, i.e., $G = E_{\mathrm{Ran}(Ge)} = E_{\mathrm{Ker}(1-G)} \leq E_{\mathbf{Ker}(1-F)} = \nu(F)$.

Theorem 2.12. *Let P be a De Morgan Poset. Then P is sharply approximating if and only if it is sharply dominating.*

Proof: "*Only if part.*" For every $a' \in P$, there exists $\nu(a') \leq a'$ since P is sharply approximating. So, $a'' \leq (\nu(a'))'$, i.e., $a \leq (\nu(a'))'$. Obviously, $(\nu(a'))' \in P_s$. For every $c \in P_s$, $a \leq c$, then, $c' \leq a'$. So, $c' \leq \nu(a')$ by $c' \in P_s$. Thus $(\nu(a'))' \leq c'' = c$. i.e., $\mu(a) = (\nu(a'))'$.

"If part." For every $a \in P$, there exists $\mu(a) \in P_s$ such that $a \leq \mu(a)$. Similarly, for a', $a' \leq \mu(a')$. Then $(\mu(a'))' \leq a'' = a$. Obviously, $(\mu(a'))' \in P_s$. For every $c \in P_s$, $c \leq a$, then $a' \leq c'$, which implies $\mu(a') \leq c'$. Thus, $c'' = c \leq (\mu(a'))'$. Hence, $\nu(a) = (\mu(a'))'$. □

Proposition 2.13. *Let P be a sharply approximating De Morgan poset. $\nu : P \longrightarrow P$ is the sharply approximating mapping, and $\mu : P \longrightarrow P$ is the sharply dominating mapping. Then (ν, μ) is a pair of adjoint.*

Proof: Obviously, for every $a, b \in P$, $a \leq b$, then $\nu(a) \leq \nu(b)$, and $\mu(a) \leq \mu(b)$. i.e., ν and μ are both monotone. For every $a \in P$, $\mu(\nu(a)) = \nu(a) \leq a$, $\nu(\mu(a)) = \mu(a) \geq a$. So, (ν, μ) is a pair of adjoint.

Clearly, ν preserves existing meet, μ preserves existing join. □

From this proposition, we see the necessity of rough approximation in *SBZ*-algebra structure (Cattaneo, 1997).

Theorem 2.14. *Let P be a sharply approximating De Morgan Poset. Then there exists a unique anti-B-complementation $*$ (resp., B-complementation $\sim$) on P such that $P_s^* = P_s (P_s^\sim = P_s)$.*

Conversely, if P is an anti-BZ- Poset (resp., BZ-Poset) in which $P_s^* = P_s$ (resp., $P_s^\sim = P_s$), then P is sharply approximating and $a^* = (\nu(a))'$ (resp., $a^\sim = \nu(a')$), for all $a \in P_s$.

Proof: (i) Let P be a sharply approximating De Morgan Poset. $\nu : P \to P$ is the sharply approximating mapping. Obviously, $\nu(1) = 1$ and $\nu(a) \leq a$. Let $a, b \in P$, if $a \leq b$, then $\nu(a) \leq \nu(b)$. Since for every $a \in P_s$, $\nu(a) = a$. Then $\nu(\nu(a)') = \nu(a)'$ by $\nu(a)' \in P_s$. Evidently, $a \vee \nu(a)' = 1$ by $\nu(a) \leq a$ and $\nu(a) \in P_s$. Hence, define $a^* = \nu(a)'$. Then $(P, \leq, 0, 1, *, ')$ is an anti-*BZ*-Poset by Proposition 2.4.

To show $P_s^* = P_s . P_s^* \subseteq P_s$ by Remark 2.6. Assume that $a \in P_s$. Then $a = \nu(a)$ such that $a' = a^*$. So $a \in P_s^*$. Thus, $P_s^* = P_s$.

For uniqueness, suppose Δ is an anti-B-complementation on P such that $P_s^\Delta = P_s$. Since $a^{\Delta\Delta} = a^{\Delta'}$ is the largest sharp element that approximating a, we have $a^{\Delta\Delta} = a^{\Delta'} \leq a$. For $b \in P_s$, $b \leq a$, then $a^\Delta \leq b^\Delta$, $b = b^{\Delta\Delta} = b^{\Delta'} \leq a^{\Delta'} = a^{\Delta\Delta}$. So $a^{\Delta'} = \nu(a)$, i.e., $a^\Delta = (\nu(a))' = a^*$. Hence, $*$ is unique.

(ii) For $a \in P$, $a^{**} \in P_s$ and $a^{**} \leq a$. If $b \in P_s$ and $b \leq a$, then $b = b^{**} \leq a^{**} \leq a$. So $a^{**} = \nu(a)$. Then P is sharply approximating and $(\nu(a))' = (a^{**})' = a^{***} = a^*$. □

For another case, we can refer to Gudder (1998a).

Corollary 2.15. *Let P be a sharply approximating De Morgan Poset. $\nu : P \longrightarrow P$ is the sharply approximating mapping. Let $a^* = \nu(a)'$, $a^\sim = \nu(a')$. Then,*

(i) $\sim * : P \longrightarrow P$ is an interior operator.
(ii) $\sim * : P \longrightarrow P$ is a closure operator.
(iii) For every $a \in P$, $a^{*\sim *} = a^*$, $a^{\sim * \sim} = a^\sim$.
(iv) For every $a \in P_s$, $a^{*\sim *} = a^{\sim * \sim} = a'$.

Definition 2.16. If $(P, 0, 1, \oplus)$ is an effect algebra and $*$ is an anti-B-complementation on P, we call $(P, 0, 1, \oplus, *)$ an anti-*BZ*-effect algebra. In particular, $(P, \leq, 0, 1, ', *)$ is an anti-*BZ*-Poset. By Theorem 2.14, we can obtain.

Corollary 2.17. *(i) If P is a sharply approximating effect algebra, then there exists a unique anti-B-complementation $*$ on P such that $(P, 0, 1, \oplus, *)$ is an anti-BZ-effect algebra and $P_s^* = P_s$.*

(ii) If P is an anti-BZ-effect algebra in which $P_s^* = P_s$, then P is sharply approximating and $a^* = (\nu(a))'$, for all $a \in P$.

Corollary 2.18. *Suppose that P is a sharply approximating effect algebra. Let $a, b \in P_s$,*

(i) If $a \wedge b$ exists, then $a \wedge b \in P_s$.
(ii) If $a \vee b$ exists, then $a \vee b \in P_s$.

Proof: By Corollary 2.17 and Corollary 2.8. □

Proposition 2.19. *Let $(P, \oplus, *, 0, 1)$ be an anti-BZ-effect algebra, then P_s^* is an orthoalgebra.*

Proof: Obviously, $0, 1 \in P_s^*$, $a \in P_s^*$ iff $a' \in P_s^*$. For $a, b \in P_s^*$, $a \perp b$, then $a, b \leq (a \oplus b)$, i.e., $(a \oplus b)^* \leq a^*, b^*$. Hence, $a = a^{*'} \leq (a \oplus b)^{*'} = (a \oplus b)^{**} \leq (a \oplus b)$, $b = b^{*'} \leq (a \oplus b)^{*'} = (a \oplus b)^{**} \leq (a \oplus b)$, thus $a \oplus b = (a \oplus b)^{**}$ by Lemma 1.8. If $a \oplus a$ exists, i.e., $a \leq a'$, then $a = 0$ by $P_s^* \subseteq P_s$. □

Corollary 2.20. *If P is a sharply approximating effect algebra, then P_s is an orthoalgebra in P.*

Proof: By Corollary 2.17. □

Definition 2.21. Let P be an effect algebra. P is central approximating if every element $a \in P$ is approximated by a largest central element $\gamma(a)$.

Remark 2.22. There exists a central approximating effect algebra but not a sharply approximating effect algebra.

For example (Riecanova, 2001), let $E = \{0, a, b, a \oplus a, b \oplus b, a \oplus b, a', b', (a \oplus a)', (b \oplus b)', (a \oplus b)', 1\}$ be an effect algebra in which $a, b, (a \oplus a)', (b \oplus b)', (a \oplus b)'$ are atoms. Moreover, $a' = a \oplus (a \oplus a)' = b \oplus (a \oplus b)'$ and $b' = a \oplus (a \oplus b)' = b \oplus (b \oplus b)'$. Further for every $x \in P$, $x \oplus x' = 1$ and $0 \oplus x = x$. Then $P_s = \{a \oplus a, b \oplus b, a \oplus b, (a \oplus a)', (b \oplus b)', (a \oplus b)', 0, 1\}$. $C(P) = \{0, 1\}$. Obviously, for element $a' \in P$, there does not exist a largest sharp element approximating it. But it is easy to check $\gamma(a) = 0$ for every $a \in P$.

Conversely, a sharply approximating effect algebra may not be a central approximating effect algebra. Indeed, if P is an orthomodular lattice, then for every element $a \in P$, $\nu(a) = a$. But $\gamma(a)$ may not equal a unless $a \in C(P)$.

Theorem 2.23. *(i) If P is a central approximating effect algebra. Then there exists a unique anti-B-complementation $*$ on P such that $(P, 0, 1, \oplus, *)$ is an anti-BZ-effect algebra and $P_s^* = C(P)$.*

(ii) If P is an anti-BZ-effect algebra in which $P_s^* = C(P)$, then P is central approximating and $a^* = (\gamma(a))'$, for all $a \in P$.

Proof: (i) Let P be a central approximating effect algebra. $\gamma : P \to P$ is the central approximating mapping. Obviously, $\gamma(1) = 1$ and $\gamma(a) \leq a$. Let $a, b \in P$, if $a \leq b$, then $\gamma(a) \leq \gamma(b)$. Since for every $a \in C(P)$, $\gamma(a) = a$, then $\gamma(\gamma(a)') = \gamma(a)'$ by $\gamma(a)' \in C(P)$. Evidently, $a \vee \gamma(a)' = 1$ by $\gamma(a) \leq a$ and $\gamma(a) \in C(P)$. Hence, define $a^* = \gamma(a)'$. Then $(P, \leq, 0, 1, *, ')$ is an anti-*BZ*-effect algebra by Proposition 2.4.

Since $a \in P_s^*$ iff $a^{**} = a$ iff $a^{**} = \gamma(\gamma(a)')' = \gamma(a) = a$ iff $a \in C(P)$. Thus, $P_s^* = C(P)$.

We can prove * is unique similar to Theorem 2.13.

(ii) We have $a^{**} \leq a$ and $a^{**} \in C(P)$. Suppose $b \in C(P)$ and $b \leq a$. Then $b = b^{**} \leq a^{**} \leq a$. Hence, $\gamma(a) = a^{**}$. □

Corollary 2.24. *If P is a central approximating effect algebra. Then P_s^* is a Boolean subalgebra.*

3. S-ANTI-*BZ*-EFFECT ALGEBRAS

Definition 3.1. A De Morgan Poset P is called an S-De Morgan Poset if it satisfies the following condition: (S) $a \wedge p$ exists for all $a \in P$, $p \in P_s$. (It follows from De Morgan's laws that $a \vee p$ exists.)

Definition 3.2. An effect algebra is called an S-effect algebra if it satisfies S condition (similarly for S-anti-*BZ*-effect algebras).

Proposition 3.3. *Let P be an S-effect algebra, and let $a \in P$, $p \in P_s$.*

(i) If $a \perp p$, then $a \vee p = a \oplus p$.
(ii) If $a' \perp p'$, then $a \wedge p = (a' \oplus p')'$.
(iii) If $a \leq p$, then $a \oplus (p \wedge a') = p$.
(iv) If $p \leq a$, then $p \vee (a \wedge p') = a$.

Proof: (i) Let $a \perp p$, then $a \leq p'$ and $a \wedge p = 0$ by $p \in P_s$. Since $a \vee p$ exists, then $a \oplus p = (a \vee p) \oplus (a \wedge p) = a \vee p$ (Greechie *et al.*, 1995).

(ii) If $a' \perp p'$, then $(a' \oplus p')' = (a' \vee p')' = a \wedge p$ by (i) and De Morgan law.

(iii) Since $a \leq p$, then $p = a \oplus (a \oplus p')' = a \oplus (a \vee p')' = a \oplus (p \wedge a')$ by (i) and the effect algebra orthomodular identity.

(iv) Since $p \leq a$, then $a = p \oplus (p \oplus a')' = p \oplus (a' \vee p)' = p \oplus (a \wedge p') = p \vee (a \wedge p')$. □

Theorem 3.4. *Let P be an S-anti-BZ-effect algebra, then P_s^* is an orthomodular lattice.*

Proof: Obviously, P_s^* is an orthomodular poset by Proposition 2.15 and Proposition 3.3(i). We only have to prove P_s^* is a sublattice under the restriction order of P. For all $a, b \in P_s^*$, $(a \wedge b)^{**} = a \wedge b$ by ** is an interior operator. Since $(a \vee b)^{**} = (a' \wedge b')'^{**} = (a^* \wedge b^*)'^{**} = (a^{*'} \vee b^{*'})^{**} = (a'^* \vee b'^*)^{**} = (a' \wedge b')^{***} = (a' \wedge b')^* = a \vee b$. So P_s^* is a sublattice. Thus, P_s^* is an orthomodular lattice. □

Corollary 3.5. (Gudder, 1998b). *Let P be an S-dominating effect algebra, then P_s is an orthomodular lattice.*

Proof: Evidently, P is sharply approximating (sharply dominating). Hence, P is an anti-*BZ*-effect with $P_s^* = P_s$ by Corollary 2.17. Then P_s is an orthomodular lattice by Theorem 3.4. □

Therefore, we can obtain the classical conclusion.

Corollary 3.6. *$P(H)$ is an orthomodular lattice.*

Proof: Since $\varepsilon(H)$ is S-dominating (Gudder, 1998b). □

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